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Study of Wave Propagation in Piezo-visco-thermo-elastic Material

Vipin Gupta[1] & Daksh Sen[2]

ABSTRACT

This study examines plane wave propagation in orthotropic piezo-visco-thermo-elastic half-space using memory-dependent derivative analysis within the three-phase lag heat model. Coupled governing equations incorporating kernel function and delay time are solved via the normal mode technique. This research is vital for enhancing the performance of advanced materials in sensors, actuators, and energy harvesters, where wave propagation and thermoelectric interactions are key factors. The results could significantly impact aerospace and mechanical engineering, particularly in developing materials that can withstand complex thermal and mechanical stresses.

Keywords: Kernel Function, Memory-dependent Derivative, Three-phase Lag, Piezo-visco-thermo-elastic

INTRODUCTION

The field of thermoelasticity has evolved significantly since its inception. Biot's [1] classical thermoelasticity theory (CTT) introduced a parabolic type differential equation, but it failed to accurately describe thermal signal velocity. This limitation led to several important advancements in the field. Lord and Shulman [2] enhanced the CTT by modifying Fourier's law with the addition of a relaxation time parameter. Subsequently, Green and Naghdi [3], [4], [5] introduced three distinct thermoelastic models (GN-I, GN-II, and GN-III) to address various heat flow problems. Tzou's [6] contribution came in the form of the dual-phase-lag (2PL) model, which incorporated phase lags into Fourier's law to account for thermal inertia and microstructural interactions. Finally, Roy Choudhuri [8] expanded the 2PL heat model into the three-phase lag (3PL) by using an additional delay time to represent the thermal displacement gradient. These progressive developments have significantly enhanced our understanding and modeling capabilities in thermoelasticity.

The field of piezoelectricity has seen significant advancements since its discovery in 1880 by the Curie brothers [9]. This phenomenon, which involves the generation of electrical charge in response to applied mechanical stress, has found applications in various domains including actuation, dynamic sensing, and intelligent structures. Abo-Dahab and colleagues [12]–[16] conducted extensive studies on reflection problems involving fractional derivatives, providing new insights into the behavior of these systems. Numerous researchers [17]–[25] have explored various aspects of plane wave behavior in piezoelectric materials, furthering our understanding of wave propagation in these systems. These studies have collectively advanced our knowledge of piezoelectric materials and their applications, paving the way for innovative developments in fields such as civil engineering and energy harvesting systems.

Wang and Li's innovative concept of memory-dependent derivatives (MDD) [26] emerged from fractional derivative analysis. Their work illuminates how sliding intervals transform a Caputo-type [27] fractional derivative into an integral derivative form, characterized by a distinctive kernel function. For a function f defined on a sliding interval $[(t-\tau), t]$, the first-order MDD is expressed as an integral, incorporating a kernel function $K(t-\xi)$ and a delay time factor $\tau > 0$. This formulation can be represented mathematically as:

$$D_{\tau} f(t) = \frac{1}{\tau}\int_{t-\tau}^{t} K_{(\Omega_1,\Omega_2)}(t-\xi) f'(\xi) d\zeta \qquad \text{...(1)}$$

[1] Department of Mathematics, Gurugram University, Gurugram, India. E-mail: vipin@gurugramuniversity.ac.in
[2] Department of Mathematics, Gurugram University, Gurugram, India. E-mail: dakshsen2@gmail.com

The D_τ operates as a nonlocal operator. When the order of differentiation approaches unity, the MDD can be interpreted as a standard derivative in its most rigorous sense. The selection of an appropriate kernel function can be tailored to suit the intricacy of the problem under consideration.

$$K_{(\Omega_1,\Omega_2)}(t-\xi)=1-\frac{2\Omega_2}{\tau}(t-\xi)+\frac{\Omega_1^2}{\tau^2}(t-\xi)^2 = \begin{cases} 1 & \text{if } \Omega_1=\Omega_2=0 \\ 1-(t-\xi)/\tau & \text{if } \Omega_1=0, \Omega_2=1/2 \\ (1-(t-\xi)/\tau)^2 & \text{if } \Omega_1=\Omega_2=1 \end{cases} \quad ...(2)$$

Viscous materials, encompassing amorphous substances, polymers, and semi-crystalline materials are crucial in diverse engineering disciplines. The Voigt model [30] stands out as a prominent macroscopic mechanical model for describing viscoelastic material behavior. It effectively demonstrates how elastic materials respond to stress, emphasizing that while deformation is time-dependent, it remains recoverable. Extensive research has been conducted on various facets of viscoelasticity and thermo-viscoelasticity, exploring issues such as [31]–[36].

The present study the propagation of plane waves in piezo-visco-thermo-elastic materials, utilizing the 3PL MDD model to account for the complex interplay between thermal, mechanical, and electrical fields. The 3PL model, known for its ability to describe non-Fourier heat conduction, is particularly well-suited to capture the intricate time-dependent phenomena inherent in piezo-visco-thermo-elastic behavior.

BASIC EQUATIONS

Following Roy Choudhuri [8], Voigt [30], Guha and Singh [37], the fundamental dynamical governing equations for anisotropic piezo-visco-thermo-elastic crystals incorporating the MDD three-phase lag model under the assumption of no heat sources, body forces, or electric forces are given as:

- Generalized Hooke's law

$$\sigma_{ij}=c_{ijrl}\,e_{rl}-\eta_{ijr}\,E_r-\beta_{ij}\,T \quad ...(3)$$

- Equation of motion

$$\sigma_{ji,j}=\rho\ddot{u}_i \quad ...(4)$$

- Electric constitutive equation

$$D_i=\varepsilon_{ij}\,E_j+\eta_{ijr}\,e_{jr}+\tau_i\,T \quad ...(5)$$

with

$$E_i=-\varphi_i \quad ...(6)$$

- Equation of electrostatics

$$D_{i,i}=0 \quad (7)$$

The 3PL heat law incorporating MDD can be expressed as

$$(1+\tau_V D_T)\,K^*_{ij}T_{ij}+(1+\tau_T D_\tau)\,K^*_{ij}T_{ij} = \left(e+\tau_q D_\tau+\frac{\tau_q^2}{2}D_\tau^2\right)\left(\rho\ddot{C}_E T+T_0(\ddot{\beta}_{ij}u_{i,j}-\ddot{\tau}_i\varphi_j)\right) \quad (8)$$

where

$$c_{ijrl}=\bar{c}_{ijrl}+c^V_{ijrl}\frac{\partial}{\partial t} \quad ...(9)$$

$$\beta_{ij}=c_{ijrl}\,a^*_{ro}\,.\,(i,j,r,l=1,2,3) \quad ...(10)$$

Nomenclature

τ_q = phase lag of heat flux
T = thermal temperature
ρ = density
β_{ij} = thermal moduli tensors
φ = electrical potential
η_{ijr}, ε_{ij} = piezothermal moduli tensors
τ_T = phase lag of the temperature gradient
$\bar{C}_{ijrl}$ = elastic stiffness tensor
E_i = electric field density
a = fractional order parameter
K^*_{ij} = heat conduction tensor
τ_i = pyroelectric constants
T_0 = reference temperature
C_E = specific heat at constant strain
K_{ij} = components of the thermal conductivity
D_i = electric displacement
σ_{ij} = components of the stress
τ_V = phase lag of thermal disp. Gradient
e_{ij} = component of strain
C^V_{ijrl} = viscoelastic constants

An over-dot signifies a time derivative, while the subscript followed by a comma denotes the partial derivative.

FORMULATION OF THE PROBLEM

This study focuses on a homogeneous, orthotropic piezo-visco-thermo-elastic medium situated in the x_1x_3-plane and influenced by MDD. The constitutive relations specific to the x_1x_3-plane are represented component-wise as:

$$\sigma_{11} = \eta_{31}\varphi_{,3} + c_{11}u_{1,1} + c_{13}u_{3,3} - \beta_1 T \quad ...(11)$$

$$\sigma_{13} = \eta_{15}\varphi_{,1} + c_{55}\left(u_{1,3} + u_{3,1}\right) \quad ...(12)$$

$$\sigma_{33} = \eta_{33}\varphi_{,3} + c_{13}u_{1,1} + c_{33}u_{3,3} - \beta_3 T \quad ...(13)$$

$$D_1 = \eta_{15}\left(u_{1,3} + u_{3,1}\right) - \varepsilon_{11}\varphi_{,1} \quad ...(14)$$

$$D_3 = \eta_{31}u_{1,1} + \eta_{33}u_{3,3} - \varepsilon_{33}\varphi_{,3} + \tau_3 T \quad ...(15)$$

By applying equations (11) to (13) to equation (4), the resulting equations of motion are derived as follows:

$$c_{11}u_{1,11} + c_{55}u_{1,33} + \left(c_{55} + c_{13}\right)u_{3,13} + \left(\eta_{31} + \eta_{15}\right)\varphi_{,13} - \beta_1 T_{,1} = \rho\ddot{u}_1 \quad ...(16)$$

$$\left(c_{55} + c_{13}\right)u_{1,13} + c_{33}u_{3,33} + c_{55}u_{3,11} + \eta_{15}\varphi_{,11} + \eta_{33}\varphi_{,33} - \beta_3 T_{,3} = \rho\ddot{u}_3 \quad ...(17)$$

Utilizing equation (16), the 3PL heat equation with MDD is derived in the x_1x_3-plane

$$\left(1 + \tau_v D_\tau\right)\left(K_1^* T_{,11} + K_3^* T_{,33}\right) + \left(1 + \tau_T D_\tau\right)\left(K_1\dot{T}_{,11} + K_3\dot{T}_{,33}\right) = \left(1 + \tau_q D_\tau + \frac{\tau_q^2}{2} D_\tau^2\right)\left(\rho C_E \ddot{T} + T_0\left(\beta_1\ddot{u}_{1,1} + \beta_3\ddot{u}_{3,3} - \tau_3\ddot{\varphi}_{,3}\right)\right) \quad ...(18)$$

Equations (14) and (15) are substituted into equation (15) to obtain the electrostatics equation as follows:

$$\left(\eta_{15} + \eta_{31}\right)u_{1,13} + \eta_{15}u_{3,11} + \eta_{33}u_{3,33} - \varepsilon_{11}\varphi_{,11} - \varepsilon_{33}\varphi_{,33} + \tau_3 T_{,3} = 0 \quad ...(19)$$

where, $\beta_{ij} = \beta_{ij}\,\delta_{ij}$, $K^*_{ij} = K^*_i\,\delta_{ij}$, $K_{ij} = K_i\,\delta_{ij}$, and i is not summed.

The following non-dimensional formulas are considered:

$$\left(x_1', x_3'\right) = \frac{\omega_1}{c_1}\left(x_1, x_3\right),\ \varphi' = \frac{\omega_1\eta_{31}}{c_1\beta_1 T_0}\varphi,$$

$$\left(t', \tau_T', \tau_q', \tau_v'\right) = \omega_1\left(t, \tau_T, \tau_q, \tau_v\right),\ T' = \frac{\beta_1 T}{\rho c_1^2},$$

$$\left(u_1', u_3'\right) = \frac{\omega_1}{c_1}\left(u_1, u_3\right),\ \left(D_1', D_3'\right) = \frac{c_{11}}{\eta_{33}\beta_1 T_0}\left(D_1, D_3\right),$$

$$\sigma_{ij}' = \frac{\sigma_{ij}}{\beta_1 T_0},\ \text{where}\ \omega_1 = \frac{\rho C_E c_1^2}{K_1},\ c_1 = \sqrt{\frac{c_{11}}{\rho}} \quad ...(20)$$

After applying dimensionless transformations to equations (16)-(19) and subsequently omitting the prime notation, we obtain the following set of equations in their non-dimensional form

$$\left(\frac{\partial^2}{\partial x_1^2} + b_{11}\frac{\partial^2}{\partial x_3^2} - b_{12}\frac{\partial^2}{\partial t^2}\right)u_1 + b_{13}\frac{\partial^2 u_3}{\partial x_1\partial x_3} + b_{14}\frac{\partial^2\varphi}{\partial x_1\partial x_3} - b_{12}\frac{\partial T}{\partial x_1} = 0 \quad ...(21)$$

$$\frac{\partial^2 u_1}{\partial x_1\partial x_3} + \left(b_{21}\frac{\partial^2}{\partial x_1^2} + b_{22}\frac{\partial^2}{\partial x_3^2} - b_{23}\frac{\partial^2}{\partial t^2}\right)u_3 + \left(b_{24}\frac{\partial^2}{\partial x_1^2} + b_{25}\frac{\partial^2}{\partial x_3^2}\right)\varphi - b_{26}\frac{\partial T}{\partial x_3} = 0 \quad ...(22)$$

$$\left(1 + \tau_v D_\tau\right)\left(b_{31}\frac{\partial^2}{\partial x_1^2} + b_{32}\frac{\partial^2}{\partial x_3^2}\right)T + \left(1 + \tau_T D_\tau\right)\left(b_{33}\frac{\partial^2}{\partial x_1^2} + b_{34}\frac{\partial^2}{\partial x_3^2}\right)\dot{T} = \left(1 + \tau_q D_\tau + \frac{\tau_q^2}{2}D_\tau^2\right)\left(\frac{\partial\ddot{u}_1}{\partial x_1} + b_{35}\frac{\partial\ddot{u}_3}{\partial x_3} - b_{36}\frac{\partial\ddot{\varphi}}{\partial x_3} + b_{37}\ddot{T}\right) \quad ...(23)$$

$$\frac{\partial^2 u_1}{\partial x_1\partial x_3} + \left(b_{41}\frac{\partial^2}{\partial x_1^2} + b_{42}\frac{\partial^2}{\partial x_3^2}\right)u_3 - \left(b_{43}\frac{\partial^2}{\partial x_1^2} + b_{44}\frac{\partial^2}{\partial x_3^2}\right)\varphi + b_{45}\frac{\partial T}{\partial x_3} = 0 \quad ...(24)$$

where,

$$b_{11} = \frac{c_{44}}{c_{11}},\ b_{12} = \frac{c_{11}}{c_{11}},\ b_{13} = \frac{c_{13} + c_{44}}{c_{11}},$$

$$b_{14} = \frac{\left(\eta_{31} + \eta_{15}\right)\beta_1 T_0}{c_{11}\eta_{31}},\ b_{21} = \frac{c_{44}}{c_{44} + c_{13}},$$

$$b_{22} = \frac{c_{33}}{c_{44} + c_{13}},\ b_{23} = \frac{c_{11}}{c_{44} + c_{13}},$$

$$b_{24} = \frac{\eta_{15}\beta_1 T_0}{\eta_{31}\left(c_{44} + c_{13}\right)},\ b_{25} = \frac{\eta_{33}\beta_1 T_0}{\eta_{31}\left(c_{44} + c_{13}\right)},$$

$$b_{26} = \frac{c_{11}\beta_3}{\left(c_{44} + c_{13}\right)\beta_1},\ b_{31} = \frac{K_1^*\rho}{T_0\beta_1^2},\ b_{32} = \frac{K_3^*\rho}{T_0\beta_1^2},$$

$$b_{33} = \frac{K_1\omega_1\rho}{T_0\beta_1^2},\ b_{34} = \frac{K_3\omega_1\rho}{T_0\beta_1^2},\ b_{35} = \frac{\beta_3}{\beta_1},$$

$$b_{36} = \frac{\tau_3 T_0}{\eta_{31}},\ b_{37} = \frac{c_{11}\rho C_E}{T_0\beta_1^2},\ b_{41} = \frac{\eta_{15}}{(\eta_{15}+\eta_{31})},$$

$$b_{42} = \frac{\eta_{33}}{(\eta_{15}+\eta_{31})},\ b_{43} = \frac{\varepsilon_{11}\beta_1 T_0}{\eta_{31}(\eta_{15}+\eta_{31})},$$

$$b_{44} = \frac{\varepsilon_{33}\beta_1 T_0}{\eta_{31}(\eta_{15}+\eta_{31})},\ b_{45} = \frac{c_{11}\tau_3}{\beta_1(\eta_{15}+\eta_{31})}.$$

SOLUTION OF THE PROBLEM

The solution can be decomposed into normal modes as follows [38]:

$$[\varphi, u_1, u_3, T](x_1, x_3, t) = [\varphi^*, u_1^*, u_3^*, T^*](x_3)\exp[\iota a(x_1 - ct)] \quad ...(25)$$

where, φ^*, u^*_1, u^*_3, and T^* are the amplitudes of the functions φ, u_1, u_3, and T, respectively.

Using equation (25) into equations (21)–(24) and using $D = d/dz$, we obtain

$$\left(A_1 + b_{11}D^2\right)u_1^* + A_2 Du_3^* + A_3 D\varphi^* - A_4 T^* = 0 \quad ...(26)$$

$$A_4^* Du_1^* + \left(b_{22}D^2 + A_5\right)u_3^* + \left(A_6 + b_{25}D^2\right)\varphi^* - b_{26}DT^* = 0 \quad ...(27)$$

$$A_7 u_1^* + A_8 Du_3^* - A_9 D\varphi^* + \begin{pmatrix} A_{10} + H_1 A_{11} + H_1 b_{32} D^2 + \\ H_2 A_{12} + H_2 A_{13} D^2 \end{pmatrix} T^* = 0 \quad ...(28)$$

$$A_4^* Du_1^* + \left(A_{14} + b_{42}D^2\right)u_3^* - \left(A_{15} + b_{44}D^2\right)\varphi^* + b_{45}DT^* = 0 \quad ...(29)$$

where,

$$A_1 = a^2\left(b_{12}c^2 - 1\right),\ A_2 = \iota a b_{13},\ A_3 = \iota a b_{14},$$

$$A_4 = \iota a b_{12},\ A_4^* = \iota a,\ A_5 = -b_{21}a^2 + b_{23}a^2c^2,$$

$$A_6 = -b_{24}a^2,\ A_7 = \iota a^3c^2,\ A_8 = b_{35}a^2c^2,$$

$$A_9 = b_{36}a^2c^2,\ A_{10} = b_{37}a^2c^2,\ A_{11} = -b_{31}a^2,$$

$$A_{12} = \iota b_{33}a^3c,\ A_{13} = -b_{34}\iota ac,\ A_{14} = -a^2 b_{41},$$

$$A_{15} = -a^2 b_{43},\ H_1 = \frac{1+\tau_v\Lambda(\tau,b)}{1+\tau_q\Lambda(\tau,b)+\tau_q^2 b\Lambda(\tau,b)/2},$$

$$H_2 = \frac{1+\tau_T\Lambda(\tau,b)}{1+\tau_q\Lambda(\tau,b)+\tau_q^2 b\Lambda(\tau,b)/2},\ b^* = -\iota ac,$$

$$\Lambda(\tau, b^*) = \frac{\left(\begin{pmatrix} -b^2\tau^2\left(\Omega_1^2 - 2\Omega_2 + 1\right) + \\ 2b^*\tau\left(\Omega_1^2 - \Omega_2\right) + 2\Omega_1^2 \end{pmatrix}\exp\left(-b^*\tau\right) \\ + b^{*2}\tau^2 - 2b^*\Omega_2\tau + 2a\Omega_1^2\right)}{b^{*2}\tau^3} \quad ...(30)$$

A non-trivial solution for equations (4.2) to (4.5) exists only if the determinant of the coefficient matrix of the preceding linear system is zero. Then

$$\left(D^8 - AD^6 + BD^4 - CD^2 + E\right)\left\{\varphi^*(x_3), u_1^*(x_3), u_3^*(x_3), T^*(x_3)\right\} = 0 \quad ...(31)$$

where, the coefficient expressions are written in the Appendix.

Equation (31) can be expressed as:

$$\left(D^2 - k_1^2\right)\left(D^2 - k_2^2\right)\left(D^2 - k_3^2\right)\left(D^2 - k_4^2\right)\left\{\varphi^*(x_3), u_1^*(x_3), u_3^*(x_3), T^*(x_3)\right\} = 0 \quad ...(32)$$

The solutions of equation (32) can be written as, which are bounded as $x_3 \to \infty$.

$$\left(u_1^*, u_3^*, \varphi^*, T^*\right) = \sum_{n=1}^{4}\left(1, L_{1n}, L_{2n}, L_{3n}\right)M_n\exp\left(-k_n x_3\right) \quad ...(33)$$

where, k_n^2, ($n = 1 - 4$) are the roots of equation (33).

Substituting equation (33) into equations (11)–(15) and performing a dimensionless analysis within the normal mode approach yields

$$\begin{pmatrix} \sigma_{11}^*, \sigma_{33}^*, \sigma_{13}^*, \\ D_1^*, D_3^* \end{pmatrix} = \sum_{n=1}^{4}\begin{pmatrix} L_{4n}, L_{5n}, L_{6n}, \\ L_{7n}, L_{8n} \end{pmatrix}\exp\left(-k_n x_3\right) \quad ...(34)$$

where L_{jn}, $n = 1 - 4$, $j = 1 - 8$ are provided in the Appendix.

BOUNDARY CONDITIONS

The following boundary conditions enable the solution of the governing equations in a half-space $(x_1, x_2, x_3 \geq 0)$

$$\sigma_{33}(x_1, 0, t) = f_1(x_1, t) = -f_1^*\exp\{\iota a(x_1 - ct)\},\quad \sigma_{13}(x_1, 0, t) = 0 \quad ...(35)$$

$$T(x_1, 0, t) = f_2(x_1, t) = f_2^* \exp\{\iota a(x_1 - ct)\} \quad ...(35)$$

$$\varphi_{,3}(x_1, 0, t) = 0 \quad ...(37)$$

where, f_1 and f_2 are functions of (x_1, t), f_1^*, f_2^* are constant, $\omega = ac$ is the wave frequency, and a be the wave number in the x_1-direction.

By inserting the proposed solutions into the boundary conditions specified in equations (35)–(37) to solve for the coefficients M_n $(n = 1 - 4)$, we obtain

$$\sum_{n=1}^{4} L_{5n} M_n = -f_1^* \quad ...(38)$$

$$\sum_{n=1}^{4} L_{6n} M_n = 0 \quad ...(39)$$

$$\sum_{n=1}^{4} L_{3n} M_n = f_2^* \quad (40)$$

$$\sum_{n=1}^{4} k_n L_{2n} M_n = 0 \quad ...(41)$$

After solving equations (38)-(41), we get

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} L_{51} & L_{52} & L_{53} & L_{54} \\ L_{61} & L_{62} & L_{63} & L_{64} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ k_1 L_{21} & k_2 L_{22} & k_3 L_{23} & k_4 L_{24} \end{bmatrix}^{-1} \begin{bmatrix} -f_1^* \\ 0 \\ f_2^* \\ 0 \end{bmatrix} \quad ...(42)$$

where, the elements of matrix L_{5i}, L_{6i}, L_{3i}, L_{2i}, $i = 1 - 4$ are provided in Appendix.

Consequently, we are able to derive exact mathematical expressions for the components of the displacement vector, the temperature distribution, the electric potential field, the stress tensor elements, and the electric displacement vector.

CONCLUSIONS

This study has examined the propagation of plane waves in an orthotropic piezo-visco-thermoelastic half-space using memory-dependent derivative analysis within the three-phase lag heat model. The research has yielded several significant findings:

1. The incorporation of the memory-dependent derivative and three-phase lag model has allowed for a more comprehensive description of the complex behavior of piezo-visco-thermoelastic materials under wave propagation.
2. The governing equations, which include crucial factors such as relaxation time, time delay, and kernel functions, have been successfully formulated and solved using the normal mode analysis technique.
3. The study has provided analytical expressions for various physical quantities, including displacement components, temperature distribution, electric potential, stress tensor elements, and electric displacement vector.
4. The analytical solutions derived in this study offer a foundation for understanding the intricate interplay between thermal, mechanical, and electrical fields in piezo-visco-thermoelastic materials.

These results contribute to the broader understanding of wave propagation in advanced materials and have potential applications in the development and optimization of sensors, actuators, and energy harvesters.

APPENDIX

$$P_1 = b_{11} b_{26},\ P_2 = b_{26} A_1 - A_4 A_4^*,\ P_3 = b_{26} A_2 - b_{24} A_4,$$

$$P_4 = -A_4 A_5,\ P_5 = b_{26} A_3 - b_{25} A_4,\ P_6 = -A_4 A_6,$$

$$P_7 = b_{11} A_{17},\ P_8 = b_{11} A_{16} + A_1 A_{17},\ P_9 = A_1 A_{16} + A_4 A_7,$$

$$P_{10} = A_2 A_{17},\ P_{11} = A_2 A_{16} + A_4 A_8,\ P_{12} = A_3 A_{17},$$

$$P_{13} = A_3 A_{16} - A_4 A_9,\ P_{14} = b_{11} b_{45},\ P_{15} = b_{45} A_1 - A_4 A_4^*,$$

$$P_{16} = b_{45} A_2 + b_{42} A_4,\ P_{17} = A_4 A_{14},$$

$$P_{18} = b_{45} A_3 - A_4 b_{44},\ P_{19} = -A_4 A_{15}$$

$$\Delta_0 = P_5 P_7 P_{16} - P_3 P_7 P_{18} + P_1 P_{10} P_{18} - P_1 P_{12} P_{16} + \\ + P_3 P_{12} P_{24} - P_5 P_{10} P_{24}$$

$$\Delta_1 = P_5 P_7 P_{17} + P_5 P_8 P_{16} - P_6 P_7 P_{16} + P_1 P_{10} P_{19} \\ - P_3 P_8 P_{18} - P_4 P_7 P_{18} - P_3 P_7 P_{19} + P_1 P_{11} P_{18} \\ - P_1 P_{12} P_{17} - P_1 P_{13} P_{16} + P_2 P_{10} P_{18} - P_2 P_{12} P_{16} \\ - P_3 P_{12} P_{15} + P_3 P_{13} P_{14} + P_4 P_{12} P_{14} - P_5 P_{10} P_{15} \\ - P_5 P_{11} P_{14} - P_6 P_{10} P_{14}$$

$$\Delta_2 = P_5 P_8 P_{17} - P_3 P_9 P_{18} - P_4 P_7 P_{19} - P_4 P_8 P_{18} \\ - P_3 P_8 P_{19} + P_5 P_9 P_{16} + P_6 P_7 P_{17} + P_6 P_8 P_{16} \\ + P_1 P_{11} P_{19} - P_1 P_{13} P_{17} + P_2 P_{10} P_{19} + P_2 P_{11} P_{18} \\ - P_2 P_{12} P_{17} - P_2 P_{13} P_{16} + P_3 P_{13} P_{15} + P_4 P_{12} P_{15} \\ + P_4 P_{13} P_{14} - P_5 P_{11} P_{15} - P_6 P_{10} P_{15} - P_6 P_{11} P_{14}$$

$$\Delta_3 = P_5P_9P_{17} - P_4P_8P_{19} - P_4P_9P_{18} - P_3P_9P_{19} + P_6P_8P_{17} + P_6P_9P_{16} + P_2P_{11}P_{19} - P_2P_{13}P_{17} + P_4P_{13}P_{15} - P_6P_{11}P_{15}$$

$$\Delta_4 = P_6P_9P_{17} - P_4P_9P_{19}$$

$$q_{1n} = -P_1k_n^3 - P_2k_n,\ q_{2n} = P_3k_n^2 + P_4,$$

$$q_{3n} = P_5k_n^2 + P_6,\ q_{4n} = P_7k_n^4 + P_8q_n^2 + P_9,$$

$$q_{5n} = -P_{10}k_n^3 - P_{11}k_n,\ q_{6n} = -P_{12}k_n^3 - P_{13}k_n$$

$$A = -\frac{\Delta_1}{\Delta_0},\ B = \frac{\Delta_2}{\Delta_0},\ C = -\frac{\Delta_3}{\Delta_0},\ E = \frac{\Delta_4}{\Delta_0}$$

$$L_{1n} = \frac{q_{3n}q_{4n} - q_{1n}q_{6n}}{q_{2n}q_{6n} - q_{3n}q_{5n}},\ L_{2n} = \frac{-q_{1n} - q_{2n}L_{1n}}{q_{3n}},$$

$$L_{3n} = \frac{A_1 + d_{11}k_n^2 - A_2k_nL_{1n} - A_3k_nL_{2n}}{A_4}$$

$$L_{4n} = \frac{c_{13}\iota a}{\beta_1T_0} - \frac{c_{33}k_nL_{1n}}{\beta_1T_0} - \frac{\eta_{33}k_nL_{2n}}{\eta_{31}} - \frac{c_{11}\beta_3L_{3n}}{\beta_1^2T_0},$$

$$L_{5n} = \frac{c_{55}(\iota aL_{1n} - k_n)}{\beta_1T_0} + \frac{\eta_{15}\iota aL_{2n}}{\eta_{31}},$$

$$L_{6n} = \frac{\iota ac_{11}}{\beta_1T_0} - \frac{c_{13}k_nL_{1n}}{\beta_1T_0} - k_nL_{2n} - \frac{L_{3n}c_{11}}{\beta_1T_0},$$

$$L_{7n} = \frac{c_{11}\eta_{15}(\iota aL_{1n} - k_n)}{\beta_1T_0\eta_{33}} - \frac{c_{11}\varepsilon_{11}\iota aL_{2n}}{\eta_{33}\eta_{31}},$$

$$L_{8n} = \frac{c_{11}}{\beta_1T_0}\left(\frac{\iota a\eta_{31}}{\eta_{33}} - k_nL_{1n} + \frac{\varepsilon_{33}\beta_1T_0k_nL_{2n}}{\eta_{31}\eta_{33}} + \frac{\tau_3c_{11}L_{3n}}{\beta_1\eta_{33}}\right),$$

$$n = 1-4$$

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